No additional cost was introduced except for the difference in generating a Cauchy random number instead of a Gaussian random number.

The paper then analyzes FEP and CEP in depth in terms of search step size and neighborhood size, and explains why FEP performs better than CEP for most benchmark problems. The theoretical analysis is supported by the additional empirical evidence in which the range of initial x values was changed. The paper shows that FEP's long jumps increase the probability of finding a near-optimum when the distance between the current search point and the optimum is large, but decrease the probability when such distance is small. The paper also investigates the relationship between the neighborhood size and the probability of finding a near-optimum in this neighborhood. Some insights on evolutionary search and optimization in general have been gained from the above analyses.

The above analyses also led to an improved FEP (IFEP) which is very simple yet effective. IFEP uses the idea of mixing search biases to mix Cauchy and Gaussian mutations. Unlike some switching algorithms which have to decide when to switch between different mutations during search, IFEP does not need to make such decision and introduces no parameters. IFEP is robust, assumes no prior knowledge of the problem to be solved, and performs at least as well as the better one of FEP and CEP for most benchmark problems. Future work on IFEP includes the comparison of IFEP with other self-adaptive algorithms such as [20] and other evolutionary algorithms using Cauchy mutation [21].

The idea of FEP and IFEP can also be applied to other evolutionary algorithms to design faster optimization algorithms [22]. For $(\mu + \lambda)$ and (μ, λ) evolutionary algorithms where $\mu < \lambda$, IFES would be particularly attractive since a parent has to generate more than one offspring. It may be beneficial if different offspring are generated by different mutations [22].

10 Appendix: Benchmark Functions

10.1 Sphere Model

$$f_1(x) = \sum_{i=1}^{30} x_i^2$$

$$-100 \le x_i \le 100, \quad \min(f_1) = f_1(0, \ldots, 0) = 0$$

10.2 Schwefel's Problem 2.22

$$f_2(x) = \sum_{i=1}^{30} |x_i| + \prod_{i=1}^{30} |x_i|$$

$$-10 \le x_i \le 10, \quad \min(f_2) = f_2(0, \ldots, 0) = 0$$

10.3 Schwefel's Problem 1.2

$$f_3(x) = \sum_{i=1}^{30} \left(\sum_{j=1}^{i} x_j \right)^2$$

$$-100 \le x_i \le 100, \quad \min(f_3) = f_3(0, \ldots, 0) = 0$$

10.4 Schwefel's Problem 2.21

$$f_4(x) = \max_i \{|x_i|, 1 \leq i \leq 30\}$$

$$-100 \leq x_i \leq 100, \quad \min(f_4) = f_4(0, \ldots, 0) = 0$$

10.5 Generalized Rosenbrock's Function

$$f_5(x) = \sum_{i=1}^{29} [100(x_{i+1} - x_i^2)^2 + (x_i - 1)^2]$$

$$-30 \leq x_i \leq 30, \quad \min(f_5) = f_5(1, \ldots, 1) = 0$$

10.6 Step Function

$$f_6(x) = \sum_{i=1}^{30} \left(\lfloor x_i + 0.5 \rfloor\right)^2$$

$$-100 \leq x_i \leq 100, \quad \min(f_6) = f_6(0, \ldots, 0) = 0$$

10.7 Quartic Function with Noise

$$f_7(x) = \sum_{i=1}^{30} i x_i^4 + random[0, 1)$$

$$-1.28 \leq x_i \leq 1.28, \quad \min(f_7) = f_7(0, \ldots, 0) = 0$$

10.8 Generalized Schwefel's Problem 2.26

$$f_8(x) = -\sum_{i=1}^{30} \left(x_i \sin\left(\sqrt{|x_i|}\right) \right)$$

$$-500 \leq x_i \leq 500, \quad \min(f_8) = f_8(420.9687, \ldots, 420.9687) = -12569.5$$

10.9 Generalized Rastrigin's Function

$$f_9(x) = \sum_{i=1}^{30} [x_i^2 - 10\cos(2\pi x_i) + 10)]$$

$$-5.12 \leq x_i \leq 5.12, \quad \min(f_9) = f_9(0, \ldots, 0) = 0$$

10.10 Ackley's Function

$$f_{10}(x) = -20 \exp\left(-0.2\sqrt{\frac{1}{30}\sum_{i=1}^{30} x_i^2}\right) - \exp\left(\frac{1}{30}\sum_{i=1}^{30} \cos 2\pi x_i\right) + 20 + e$$

$$-32 \leq x_i \leq 32, \quad \min(f_{10}) = f_{10}(0, \ldots, 0) = 0$$

10.11 Generalized Griewank Function

$$f_{11}(x) = \frac{1}{4000}\sum_{i=1}^{30} x_i^2 - \prod_{i=1}^{30} \cos\left(\frac{x_i}{\sqrt{i}}\right) + 1$$

$$-600 \le x_i \le 600, \quad \min(f_{11}) = f_{11}(0,\dots,0) = 0$$

10.12 Generalized Penalized Functions

$$\begin{aligned} f_{12}(x) &= \frac{\pi}{30}\left\{10\sin^2(\pi y_1) + \sum_{i=1}^{29}(y_i-1)^2[1+10\sin^2(\pi y_{i+1})] + (y_n-1)^2\right\} \\ &\quad + \sum_{i=1}^{30} u(x_i, 10, 100, 4) \end{aligned}$$

$$-50 \le x_i \le 50, \quad \min(f_{12}) = f_{12}(1,\dots,1) = 0$$

$$\begin{aligned} f_{13}(x) &= 0.1\left\{\sin^2(\pi 3x_1 + \sum_{i=1}^{29}(x_i-1)^2[1+\sin^2(3\pi x_{i+1})] + (x_n-1)^2[1+\sin^2(2\pi x_{30})]\right\} \\ &\quad + \sum_{i=1}^{30} u(x_i, 5, 100, 4) \end{aligned}$$

$$-50 \le x_i \le 50, \quad \min(f_{13}) = f_{13}(1,\dots,1) = 0$$

where

$$u(x_i, a, k, m) = \begin{cases} k(x_i - a)^m, & x_i > a, \\ 0, & -a \le x_i \le a, \\ k(-x_i - a)^m, & x_i < -a. \end{cases}$$

$$y_i = 1 + \frac{1}{4}(x_i + 1)$$

10.13 Shekel's Foxholes Function

$$f_{14}(x) = \left[\frac{1}{500} + \sum_{j=1}^{25}\frac{1}{j + \sum_{i=1}^{2}(x_i - a_{ij})^6}\right]^{-1}$$

$$-65.536 \le x_i \le 65.536, \quad \min(f_{14}) = f_{14}(-32,-32) \approx 1$$

where

$$(a_{ij}) = \begin{pmatrix} -32 & -16 & 0 & 16 & 32 & -32 & \cdots & 0 & 16 & 32 \\ -32 & -32 & -32 & -32 & -32 & -16 & \cdots & 32 & 32 & 32 \end{pmatrix}$$

10.14 Kowalik's Function

$$f_{15}(x) = \sum_{i=1}^{11}\left[a_i - \frac{x_1(b_i^2 + b_i x_2)}{b_i^2 + b_i x_3 + x_4}\right]^2$$

$$-5 \le x_i \le 5, \quad \min(f_{15}) \approx f_{15}(0.1928, 0.1908, 0.1231, 0.1358) \approx 0.0003075$$

Table 11: Kowalik's Function f_{15}

i	a_i	b_i^{-1}
1	0.1957	0.25
2	0.1947	0.5
3	0.1735	1
4	0.1600	2
5	0.0844	4
6	0.0627	6
7	0.0456	8
8	0.0342	10
9	0.0323	12
10	0.0235	14
11	0.0246	16

10.15 Six-hump Camel-Back Function

$$f_{16} = 4x_1^2 - 2.1x_1^4 + \frac{1}{3}x_1^6 + x_1x_2 - 4x_2^2 + 4x_2^4$$
$$-5 \le x_i \le 5$$
$$x_{min} = (0.08983, -0.7126), \ \ (-0.08983, 0.7126)$$
$$\min(f_{16}) = -1.0316285$$

10.16 Branin Function

$$f_{17}(x) = \left(x_2 - \frac{5.1}{4\pi^2}x_1^2 + \frac{5}{\pi}x_1 - 6\right)^2 + 10\left(1 - \frac{1}{8\pi}\right)\cos x_1 + 10$$
$$-5 \le x_1 \le 10, \quad 0 \le x_2 \le 15$$
$$x_{min} = (-3.142, 12.275), \ \ (3.142, 2.275), \ \ (9.425, 2.425)$$
$$\min(f_{17}) = 0.398$$

10.17 Goldstein-Price Function

$$\begin{aligned} f_{18}(x) \ &= \ [1 + (x_1 + x_2 + 1)^2(19 - 14x_1 + 3x_1^2 - 14x_2 + 6x_1x_2 + 3x_2^2)] \\ &\quad \times[30 + (2x_1 - 3x_2)^2(18 - 32x_1 + 12x_1^2 + 48x_2 - 36x_1x_2 + 27x_2^2)] \end{aligned}$$
$$-2 \le x_i \le 2, \quad \min(f_{18}) = f_{18}(0, -1) = 3$$

10.18 Hartman's Family

$$f(x) = -\sum_{i=1}^{4} c_i \exp\left[-\sum_{j=1}^{n} a_{ij}(x_j - p_{ij})^2\right]$$

with $n = 3, 6$ for $f_{19}(x)$ and $f_{20}(x)$, respectively, $0 \le x_j \le 1$. The coefficients are defined by Tables 12 and 13, respectively.

For $f_{19}(x)$ the global minimum is equal to -3.86 and it is reached at the point $(0.114, 0.556, 0.852)$. For $f_{20}(x)$ the global minimum is -3.32 at the point $(0.201, 0.150, 0.477, 0.275, 0.311, 0.657)$.

Table 12: Hartman Function f_{19}

i	$a_{ij}, j=1,2,3$			c_i	$p_{ij}, j=1,2,3$		
1	3	10	30	1	0.3689	0.1170	0.2673
2	0.1	10	35	1.2	0.4699	0.4387	0.7470
3	3	10	30	3	0.1091	0.8732	0.5547
4	0.1	10	35	3.2	0.038150	0.5743	0.8828

Table 13: Hartman Function f_{20}

i	$a_{ij}, j=1,\cdots,6$						c_i	$p_{ij}, j=1,\cdots,6$					
1	10	3	17	3.5	1.7	8	1	0.1312	0.1696	0.5569	0.0124	0.8283	0.5886
2	0.05	10	17	0.1	8	14	1.2	0.2329	0.4135	0.8307	0.3736	0.1004	0.9991
3	3	3.5	1.7	10	17	8	3	0.2348	0.1415	0.3522	0.2883	0.3047	0.6650
4	17	8	0.05	10	0.1	14	3.2	0.4047	0.8828	0.8732	0.5743	0.1091	0.0381

10.19 Shekel's Family

$$f(x) = -\sum_{i=1}^{m}[(x-a_i)(x-a_i)^T + c_i]^{-1}$$

with $m = 5, 7$ and 10 for $f_{21}(x)$, $f_{22}(x)$ and $f_{23}(x)$, respectively, $0 \leq x_j \leq 10$.

Table 14: Shekel Functions f_{21}, f_{22}, f_{23}

i	$a_{ij}, j=1,\cdots,4$				c_i
1	4	4	4	4	0.1
2	1	1	1	1	0.2
3	8	8	8	8	0.2
4	6	6	6	6	0.4
5	3	7	3	7	0.4
6	2	9	2	9	0.6
7	5	5	3	3	0.3
8	8	1	8	1	0.7
9	6	2	6	2	0.5
10	7	3.6	7	3.6	0.5

These functions have 5, 7 and 10 local minima for $f_{21}(x)$, $f_{22}(x)$, and $f_{23}(x)$, respectively. $x_{local\_opt} \approx a_i$, $f(x_{local\_opt}) \approx 1/c_i$ for $1 \leq i \leq m$. The coefficients are defined by Table 14.

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